Finite Math - Spring 2017
Lecture Notes - 4/7/2017

Homework

- Section 4.5 - 19, 20, 25, 26, 29, 32, 33, 35, 39, 42, 45, 51, 55, 61, 64, 77, 78, 79, 80

Section 4.5 - Inverse of a Square Matrix

Example 1. *Find the inverse of the matrix*

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & -3 \\ -2 & -3 & -1 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 & 1 & 0 \\ -2 & -3 & -1 & 0 & 0 & 1 \end{array}\right] \overset{R_1\leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ -2 & -3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3+R_2\to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right] \overset{R_2-2R_1\to R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & -2 & 6 & 1 & -2 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\overset{R_2-3R_3\to R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right] \overset{R_3+R_2\to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 8 & -1 & -2 & -2 \end{array}\right]$$

$$\overset{R_1-2R_2\to R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -21 & 4 & 5 & 6 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 8 & -1 & -2 & -2 \end{array}\right] \overset{\frac{1}{8}R_3\to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -21 & 4 & 5 & 6 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$$\overset{R_1+21R_3\to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{8} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \overset{R_1+21R_3\to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{8} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 1 & -\frac{7}{8} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

Example 2. *Find the inverse of the matrix:*

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}.$$

Cryptography. Suppose we represent letters by numbers as follows

Blank	0	I	9	R	18
A	1	J	10	S	19
B	2	K	11	T	20
C	3	L	12	U	21
D	4	M	13	V	22
E	5	N	14	W	23
F	6	O	15	X	24
G	7	P	16	Y	25
H	8	Q	17	Z	26

Then, for example, the message "SECRET CODE" would correspond to the sequence

19 5 3 18 5 20 0 3 15 4 5

The goal of Cryptography is to encode messages in a different sequence which can only be translated back to the message using a decoder.

Definition 1 (Encoding matrix/Decoding matrix)**.** *Any matrix with positive integer elements whose inverse exists can be used as an* encoding matrix*. The inverse of an encoding matrix is a* decoding matrix*.*

To encode a message, we must first decide on a encoding matrix A. If A is a $n \times n$ matrix, then we create another matrix $n \times p$ matrix B by entering the message going down columns and taking as many columns as necessary to fit the whole message. Note that the number of rows of B MUST MATCH the size of A. If there are extra entries in B after fitting the whole message, just fill them with 0's.

Example 3. *Encode the message "SECRET CODE" using the encoding matrix*

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

Solution. *We first make the matrix B*

$$B = \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix}$$

Then to encode the message we find the product

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 38+15 & 6+54 & 10+60 & 0+9 & 30+12 & 10+0 \\ 19+5 & 3+18 & 5+20 & 0+3 & 15+4 & 5+0 \end{bmatrix} \\ &= \begin{bmatrix} 53 & 60 & 70 & 9 & 42 & 10 \\ 24 & 21 & 25 & 3 & 19 & 5 \end{bmatrix} \end{aligned}$$

So the coded message is

$$53\ 24\ 60\ 21\ 70\ 25\ 9\ 3\ 42\ 19\ 10\ 5$$

Example 4. *A message was encoded with A from the previous example. Decode the sequence*

$$29\ 12\ 69\ 28\ 70\ 25\ 111\ 43$$

Solution. *First we have to invert the encoding matrix to get the decoding matrix*

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

Make a matrix out of the coded message in the same way as above

$$C = \begin{bmatrix} 29 & 69 & 70 & 111 \\ 12 & 28 & 25 & 43 \end{bmatrix}$$

and find the product

$$\begin{aligned} A^{-1}C &= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 29 & 69 & 70 & 111 \\ 12 & 28 & 25 & 43 \end{bmatrix} \\ &= \begin{bmatrix} -29+36 & -69+84 & -70+75 & -111+129 \\ 29-24 & 69-56 & 70-50 & 111-86 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 15 & 5 & 18 \\ 5 & 13 & 20 & 25 \end{bmatrix} \end{aligned}$$

Decoded, the message is

$$7\ 5\ 15\ 13\ 5\ 20\ 18\ 25$$

which, back in letters, is

GEOMETRY

Example 5. *Use the encoding matrix*

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}.$$

(a) *Encode the message "MATH IS FUN" using E.*

(b) *Decode the sequence*

$$39\ 60\ 91\ 65\ 110\ 125\ 6\ 7\ 16\ 44\ 63\ 113\ 37\ 53\ 87$$